Continuity of relative entropy of entanglement

Matthew J. Donald[1,*] and Michał Horodecki[2,**]

[1] *The Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, Great Britain*

[2] *Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80–952 Gdańsk, Poland*

We show that an entanglement measure called relative entropy of entanglement satisfies a strong continuity condition. If two states are close to each other then so are their entanglements per particle pair in this measure. It follows in particular, that the measure is appropriate for the description of entanglement manipulations in the limit of an infinite number of pairs of particles.

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Entanglement is a crucial parameter in modern quantum information theory [1–6]. It is therefore desirable to investigate properties of the functions that quantify entanglement (entanglement measures) [6–9]. A property that has recently appeared to be an important characteristic of entanglement measures is continuity [10,11]. This is especially relevant in the description of manipulations of entanglement in the regime of large numbers of identically prepared entangled pairs (that is, in the case of stationary, memoryless sources) as for example in the case of distillation of entanglement [5]. In general, one is interested in the conversion of m pairs of particles, with each pair in state ϱ, into n pairs in another state ϱ' by means of local quantum operations and classical communication (LQCC) [6]. Of course perfect transformation

$$\varrho^{\otimes m} \to \varrho'^{\otimes n}$$

is usually impossible. Thus one permits imperfections and requires only asymptotically perfect transformations: the state $\varrho^{\otimes m}$ is transformed into some state ϱ'_n, that for large n approaches $\varrho^{\otimes n}$. In this case one is interested in entanglement measures that attribute approximately the same entanglement per pair both to ϱ'_n and $\varrho^{\otimes n}$

$$D(\varrho'^{\otimes n}, \varrho'_n) \stackrel{n\to\infty}{\to} 0 \quad \Rightarrow \quad \frac{1}{n}|E(\varrho'^{\otimes n}) - E(\varrho'_n)| \stackrel{n\to\infty}{\to} 0 \tag{1}$$

where D is a chosen metric. We speak here about entanglement *per pair* (or entanglement *density*) because, in the limit of infinite numbers of pairs, one must use intensive quantities [7] (as in the thermodynamics of lattice systems). It appears that the above continuity, even if imposed only for pure ϱ', puts severe constraints on entanglement measures: all the additive measures satisfying that condition must coincide on pure states [7,10] and must be confined between entanglement of distillation and entanglement of formation [6] for mixed states [11].

In this paper, we consider the very important entanglement measure: relative entropy of entanglement [8,9]. For a state σ acting on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, this is given by

$$E_r(\sigma) = \inf_{\varrho\in\mathcal{D}} S(\sigma|\varrho)$$

where $S(\sigma|\varrho) = \mathrm{tr}\,\sigma\log\sigma - \mathrm{tr}\,\sigma\log\varrho$ and $\mathcal{D}$ is the set of separable (disentangled) states. This measure was proved [9,12] to be a tight bound for distillable entanglement, the central parameter of entanglement based quantum communication [6] (for the most straightforward proof, see [11]). We show that it satisfies the very strong continuity requirement constituted by the Fannes-type inequality (proved originally for the von Neumann entropy [13])

$$|E_r(\varrho) - E_r(\sigma)| \leq B\log\dim\mathcal{H}\,||\varrho-\sigma|| + C\eta(||\varrho-\sigma||) \tag{2}$$

where ϱ and σ act on the Hilbert space $\mathcal{H}$, B and C are constants, $\eta(s) = -s\log s$, and we use the trace norm as a metric on states: $D(\varrho,\sigma) = ||\varrho-\sigma|| \equiv \mathrm{tr}\,|\varrho-\sigma|$. A similar inequality was obtained recently by Nielsen [14] for another entanglement measure: entanglement of formation. (Nielsen uses the Bures metric as a measure of distance.)

Inequality (2) shows that relative entropy of entanglement has very regular asymptotic behaviour, and is a suitable parameter to describe asymptotic manipulations of entanglement. In particular, it is easy to see that the inequality guarantees the continuity of the form (1). Our proof is also valid for variations of the considered measure (for example, if, as in [12], we minimize over positive partial transpose states rather than over separable states). Note here that it is not clear whether one should expect such strong continuity for all relevant parameters in quantum entanglement theory. For example, distillable entanglement satisfies the weaker continuity (1) for pure ϱ', but might not satisfy the inequality (2) e.g. near the border between bound entangled and free entangled states [15,16]. It should also be noted that, even on finite dimensional Hilbert spaces, relative entropy itself is not a continuous function so that continuity of related functions cannot be taken for granted. Relative entropy is however lower semicontinuous [17,18] — on convergent sequences of states, it can jump down but not up.

Theorem *Let $\mathcal{D}$ be a set of states (density matrices) on a Hilbert space $\mathcal{H}$ of dimension $N < \infty$. Suppose that $\mathcal{D}$ is a compact convex set which includes the maximally chaotic state $\tau \equiv \frac{I}{N}$. Then the function given by*

$$E(\sigma) = \inf_{\varrho \in \mathcal{D}} S(\sigma|\varrho), \quad \textit{where} \quad S(\sigma|\varrho) = \operatorname{tr}\sigma\log\sigma - \operatorname{tr}\sigma\log\varrho$$

satisfies the inequality

$$|E(\sigma_1) - E(\sigma_2)| \leq 2(||\sigma_1 - \sigma_2||\log N + \eta(||\sigma_1 - \sigma_2||)) + 4||\sigma_1 - \sigma_2|| \tag{3}$$

for $||\sigma_1 - \sigma_2|| \leq \frac{1}{3}$.

Remarks (i) If $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{D}$ is the set of separable (disentangled) states, then E is the relative entropy of entanglement. If, instead, $\mathcal{D}$ is the set of matrices with positive partial transposition [19,16] then we obtain Rains [12] bound for distillable entanglement. (ii) Note that the inequality (3) can be written in the form (2).

Proof For any state σ, let $\hat{\varrho}(\sigma) \in \mathcal{D}$ denote a state such that $E(\sigma) = S(\sigma|\hat{\varrho}(\sigma))$. $\hat{\varrho}(\sigma)$ exists because $\mathcal{D}$ is compact and S is lower semicontinuous. Let $S_1(\sigma) = -\operatorname{tr}(\sigma\log\sigma)$ be von Neumann entropy. The theorem is clearly true if $\sigma_1 = \sigma_2$, so suppose otherwise. For $0 < x \leq 1$, let $E_x(\sigma) = \inf\{S(\sigma|x\varrho + (1-x)\tau) : \varrho \in \mathcal{D}\}$. Choose $\hat{\varrho}_x(\sigma) \in \mathcal{D}$ such that $E_x(\sigma) = S(\sigma|x\hat{\varrho}_x(\sigma) + (1-x)\tau)$. By the monotonicity of the logarithm,

$$\begin{aligned} S(\sigma|x\varrho + (1-x)\tau) &= -S_1(\sigma) - \operatorname{tr}(\sigma\log(x\varrho + (1-x)\tau)) \\ &\leq -S_1(\sigma) - \operatorname{tr}(\sigma\log\varrho) - \log x = S(\sigma|\varrho) - \log x. \end{aligned}$$

Thus

$$E(\sigma) - \log x = S(\sigma|\hat{\varrho}(\sigma)) - \log x \geq S(\sigma|x\hat{\varrho}(\sigma) + (1-x)\tau) \geq E_x(\sigma) \geq E(\sigma).$$

Set $x = 1 - ||\sigma_1 - \sigma_2||$. By the simple inequality $|\log x| \leq 2(1-x)$ for $\frac{1}{2} \leq x \leq 1$, we have $|\log x| \leq 2||\sigma_1 - \sigma_2||$ so that $|E(\sigma) - E_x(\sigma)| \leq 2||\sigma_1 - \sigma_2||$. Now we shall use Fannes' inequality [13]

$$|S_1(\sigma_1) - S_1(\sigma_2)| \leq ||\sigma_1 - \sigma_2||\log N + \eta(||\sigma_1 - \sigma_2||) \tag{4}$$

which holds for $||\sigma_1 - \sigma_2|| \leq \frac{1}{3}$. The monotonicity of the logarithm gives

$$0 \geq \log(x\varrho + (1-x)\tau) \geq -\log N + \log(1-x).$$

Composing this with the standard inequality [20]

$$|\operatorname{tr}(\sigma_1 A) - \operatorname{tr}(\sigma_2 A)| \leq ||\sigma_1 - \sigma_2||\ ||A||_{op}$$

(where $||\cdot||_{op}$ denotes operator norm), which holds for any operators σ_1, σ_2 and A in finite dimensions, we obtain

$$\begin{aligned} &|\operatorname{tr}(\sigma_1\log(x\varrho + (1-x)\tau)) - \operatorname{tr}(\sigma_2\log(x\varrho + (1-x)\tau))| \\ &\quad \leq ||\sigma_1 - \sigma_2||\,||\log(x\varrho + (1-x)\tau)||_{op} \\ &\quad \leq ||\sigma_1 - \sigma_2||(\log N - \log(1-x)) = ||\sigma_1 - \sigma_2||\log N + \eta(||\sigma_1 - \sigma_2||). \end{aligned} \tag{5}$$

From the inequalities (4) and (5) it follows that

$$|S(\sigma_1|x\varrho + (1-x)\tau) - S(\sigma_2|x\varrho + (1-x)\tau)| \leq 2(||\sigma_1 - \sigma_2||\log N + \eta(||\sigma_1 - \sigma_2||)).$$

Then

$$\begin{aligned} E_x(\sigma_1) &= S(\sigma_1|x\hat{\varrho}_x(\sigma_1) + (1-x)\tau) \leq S(\sigma_1|x\hat{\varrho}_x(\sigma_2) + (1-x)\tau) \\ &\leq S(\sigma_2|x\hat{\varrho}_x(\sigma_2) + (1-x)\tau) + 2(||\sigma_1 - \sigma_2||\log N + \eta(||\sigma_1 - \sigma_2||)) \\ &= E_x(\sigma_2) + 2(||\sigma_1 - \sigma_2||\log N + \eta(||\sigma_1 - \sigma_2||)) \end{aligned}$$

and, by symmetry

$$|E_x(\sigma_1) - E_x(\sigma_2)| \le 2(||\sigma_1 - \sigma_2|| \log N + \eta(||\sigma_1 - \sigma_2||)).$$

Finally,

$$\begin{aligned} |E(\sigma_1) - E(\sigma_2)| &\le |E(\sigma_1) - E_x(\sigma_1)| + |E_x(\sigma_1) - E_x(\sigma_2)| + |E_x(\sigma_2) - E(\sigma_2)| \\ &\le 2(||\sigma_1 - \sigma_2|| \log N + \eta(||\sigma_1 - \sigma_2||)) + 4||\sigma_1 - \sigma_2||. \end{aligned}$$

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It is often suggested that we interpret $\hat{\varrho}(\sigma)$ as the state in $\mathcal{D}$ closest to σ. This suggestion is not entirely unproblematic. For example, even if $\mathcal{D}$ is the separable states, there are σ for which $\hat{\varrho}(\sigma)$ is not unique. As a final result, therefore, we use our theorem and standard results about relative entropy ([17,18]) to prove a non-obvious property of $\hat{\varrho}(\sigma)$, which is essential to this interpretation.

Corollary *Let σ_n be a sequence of states converging to a state σ which is in $\mathcal{D}$. Then $\hat{\varrho}(\sigma_n)$ also converges to σ.*

Proof Suppose not. By compactness, there is a subsequence such that $\hat{\varrho}(\sigma_{n_k}) \to \varrho \neq \sigma$. By the theorem $E(\sigma_n) \to 0$. But

$$\liminf E(\sigma_{n_k}) = \liminf S(\sigma_{n_k}|\hat{\varrho}(\sigma_{n_k})) \ge S(\sigma|\varrho) > 0.$$ ∎

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* E-mail address: matthew.donald@phy.cam.ac.uk
** E-mail address: michalh@iftia.univ.gda.pl

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